The quantum group $SU_q(2)$, quasitriangularity, and application to the q-rotator model

Paul Bracken

Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1

Received 6 March 1996

The quantum group structure of $SU_q(2)$ is described. The property of quasitriangularity and the Yang–Baxter equation are reviewed. A universal R-matrix for this algebra is written down. It is then shown in detail that this R-matrix satisfies the triangularity equations of Drinfeld and the Yang–Baxter equation given the algebraic $SU_q(2)$ commutation relations. In physical terms, the group can be realized as the q-rotator. A specific physical application to diatomic molecules is presented.

1. Introduction

Quantum groups have assumed an important role in the description of many physical theories and have generated many interesting applications, as several recent papers have indicated [1,2]. It has even been found possible to find quantum group symmetries in classical systems. The term originates with Drinfeld [3] and has come to mean certain special Hopf algebras which are nontrivial deformations of the enveloping Hopf algebras of semi-simple Lie algebras. It is well known that the quantum systems which are described by quantum symmetries reduce to the quantum systems described by Lie symmetries when the deformation parameter approaches unity. A recent introduction to the uses of quantum groups in physical systems can be found in [4,5].

It is the purpose here to present an introduction to the Hopf algebra structure of the quantum group $SU_q(2)$ and a simple introduction to the Yang–Baxter equation, with a simple proof in the case of quasi-triangularity. It can be applied to the quantum group $SU_q(2)$ and a set of Hopf relations will be given. It is then the intent to show that an R-matrix can be written down which satisfies the required properties, as well as the Yang–Baxter equation. These ideas lead to powerful techniques; for example, the concept of braided bialgebras, which is due to Drinfeld, provides a systematic method of producing solutions of the Yang–Baxter equation [6]. Although a relatively straightforward algebra is introduced here, it is hoped that the great detail that is provided will fill a gap in the literature.

Finally, an application of these ideas to a particular system of physical interest will be presented. The q-rotator model will be described. This turns out to be an exactly solvable system. It is the q-deformation of the rigid rotator model with the quantum group symmetry $SU_q(2)$. The energy of the system is given as the eigenvalue of the Casimir operator in terms of the deformation parameter. For a particular diatomic molecule, the frequency of emission-absorption can be evaluated and fit as a function of the rotational quantum number J. In fact, this model can be understood as a nonrigid rotator, and the deformation parameter is the very quantity which characterizes the nonrigidity.

2. Algebraic structure of the quantum group

In order to provide an introduction, and to motivate what follows, let us introduce some standard terminology [6,8,9]. The area is extremely rich in terms of concepts and information, and we will only give what is essential for this article. A vector space H that is endowed with a multiplication $\mu : H \otimes H \rightarrow H$, and a unit $\eta : F \rightarrow H$ is called an algebra with unit if and only if

$$\eta(1)h = \mu(1 \otimes h) = \mu(h \otimes 1) = h\eta(1)$$

for all $h \in H$ and $\eta(1) = I$, with I the identity on H, the sequence of mappings $I \otimes \mu$, μ commutes with $\mu \otimes I$, μ, and that the element $\eta(1)$ of H is a left and a right unit for μ. Dualizing the above definition, the co-algebra with coproduct is obtained. There is a compatibility condition between these two structures but it need not concern us here.

A bialgebra is a quintuple $(H, \mu, \eta, \Delta, \epsilon)$ where (H, μ, η) is an algebra and (H, Δ, ϵ) is a coalgebra which verify a set of equivalence conditions. Given an algebra (H, μ, η) and a coalgebra (C, Δ, ϵ) a bilinear map will be defined, the convolution on the vector space of linear maps from C to H:

$$(f * g)(x) = \sum_{(x)} f(x')g(x'')$$

for any element $x \in C$. An element γ of the space of linear maps on A is called an anti-pode if γ is inverse to I under the convolution. A Hopf algebra is a bialgebra with an antipode. A morphism of Hopf algebras is a linear mapping between the underlying bialgebras commuting with the antipodes.

The quantum group $SU_q(2)$ is the q-deformation of the Lie algebra su(2) with the following algebraic commutation relations:

$$[J_q^+, J_q^-] = [2J_q^3]_q ,$$

$$[J_q^3, J_q^\pm] = \pm J_q^\pm . \qquad (1)$$

The Hopf operations will be introduced on this algebra, and then they will be used to construct and manipulate an appropriate R-matrix for this algebra. To do so, let us specify the Hopf operations. First of all, the coproduct will be defined in terms of the generators as follows:

$$\Delta(J_q^3) = J_q^3 \otimes 1 + 1 \otimes J_q^3 ,$$
$$\Delta(J_q^+) = J_q^+ \otimes q^{2J_q^3} + 1 \otimes J_q^+ , \tag{2}$$
$$\Delta(J_q^-) = J_q^- \otimes 1 + q^{-2J_q^3} \otimes J_q^- .$$

The antipodal mapping acts as follows:

$$S(J_q^3) = -J_q^3 , \quad S(J_q^\pm) = -q^{\pm 1} J_q^\pm , \tag{3}$$

and the co-unit as

$$\epsilon(J_q^\pm) = \epsilon(J_q^3) = 0 , \quad \epsilon(1) = 1 . \tag{4}$$

Following Drinfeld, one says a Hopf algebra A is quasi-triangular if there exists an invertible element $R \in A \otimes A$ such that

$$\sigma \circ \Delta R = R\Delta , \tag{5.1}$$
$$(id \otimes \Delta)R = R_{13} \cdot R_{12} , \tag{5.2}$$
$$(\Delta \otimes id)R = R_{13} \cdot R_{23} . \tag{5.3}$$

The following notation has been used. Given any element $V \in A \otimes A$ which can be written as $V = \Sigma_i a_i \otimes b_i$, then,

$$V_{12} = \Sigma_i a_i \otimes b_i \otimes 1 , \quad V_{13} = \Sigma_i a_i \otimes 1 \otimes b_i , \quad V_{23} = \Sigma_i 1 \otimes a_i \otimes b_i . \tag{6}$$

The operator σ which occurs in (5) is defined by $\sigma(x \otimes y) = y \otimes x$.

3. Quasitriangularity of the algebra

The importance of quasi-triangular Hopf algebras is that the canonical element R satisfies the quantum Yang–Baxter equation:

$$R_{12} \cdot R_{13} \cdot R_{23} = R_{23} \cdot R_{13} \cdot R_{12} . \tag{7}$$

This can be verified using the equations in (5) in the following way. The R-matrix can be written for the purposes here as $R = \sum_i a_i \otimes b_i$. It then follows that

$$\begin{aligned} R_{13}R_{23} = (\Delta \otimes I)R = \sum_i \Delta(a_i) \otimes b_i &= \sum_i (R^{-1}\Delta^T(a_i)R) \otimes b_i \\ &= R_{12}^{-1}\left(\sum_i \Delta^T(a_i) \otimes b_i\right) R_{12} \\ &= R_{12}^{-1} R_{23} R_{13} R_{12} . \end{aligned}$$

An R-matrix will be written down for this algebra, and it will be shown in detail that the quasitriangularity conditions are satisfied. In terms of the J operators, the universal R-matrix will take the following form:

$$R = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(1-q^{-2})^n}{[n]_q!} q^{2(J_q^3 \otimes J_q^3)} (J_q^+)^n \otimes (J_q^-)^n . \tag{8}$$

This will be shown systematically by verifying each of the equations in (5) separately. To proceed, a number of lemmas are required which will be referred to repeatedly, as well as the following notation:

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} ,$$

$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q \cdot [1]_q ,$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q! [n-m]_q!} .$$

These are rational functions of q over Q. It is also found useful to introduce q as an exponential, that is $q = e^h$. Both notations will be used. These lemmas, which are required to finish the proofs, are collected and proved in a group in the appendix. However, one which will be used frequently will be presented here.

LEMMA 1

Suppose the elements X, Y and Z satisfy $[X, Y] = YZ$ and $[X, Z] = 0$, then

$$e^{hX} Y e^{-hX} = Y e^{hZ} .$$

There is an elegant demonstration of this. The first commutator implies $XY = YX + YZ = Y(X+Z)$. By induction one has $X^n Y = Y(X+Z)^n$. Therefore, if $[X, Z] = 0$,

$$e^{hX} Y = \sum_{n=0}^{\infty} \frac{h^n}{n!} X^n Y = \sum_{n=0}^{\infty} \frac{h^n}{n!} (X+Z)^n = Y e^{h(X+Z)} = Y e^{hX} e^{hZ} .$$

To show that (8) satisfies (5.1), let us define

$$A_n = \frac{q^{n(n+1)/2}(1-q^{-2})^n}{[n]_q!} .$$

Applying σ to Δ, the left-hand side of eq. (5.1) becomes

$$(J_q^+ \otimes 1 + e^{2hJ_q^3} \otimes J_q^+) \cdot \sum_{n=0}^{\infty} A_n e^{2h(J_q^3 \otimes J_q^3)} (J_q^+)^n \otimes (J_q^-)^n$$

$$= \sum_{n=0}^{\infty} A_n ((J_q^+ \otimes 1) e^{2h(J_q^3 \otimes J_q^3)} (J_q^+)^n \otimes (J_q^-)^n + \left(e^{2hJ_q^3} \otimes J_q^+\right) e^{2h(J_q^3 \otimes J_q^3)} (J_q^+)^n \otimes (J_q^-)^n)$$

$$= \sum_{n=1}^{\infty} A_n (e^{2h(J_q^3 \otimes J_q^3)} e^{-2h(J_q^3 \otimes J_q^3)} (J_q^+ \otimes 1) e^{2h(J_q^3 \otimes J_q^3)} (J_q^+)^n \otimes (J_q^-)^n$$

$$+ e^{2h(J_q^3 \otimes J_q^3)} e^{-2h(J_q^3 \otimes J_q^3)} (e^{2hJ_q^3} \otimes J_q^+) e^{2h(J_q^3 \otimes J_q^3)} (J_q^+)^n \otimes (J_q^-)^n$$

$$= \sum_{n=0}^{\infty} A_n (e^{2h(J_q^3 \otimes J_q^3)} (J_q^+ \otimes 1) \cdot (1 \otimes e^{-2hJ_q^3}) (J_q^+)^n \otimes (J_q^-)^n$$

$$+ e^{2h(J_q^3 \otimes J_q^3)} (1 + J_q^+) (J_q^+)^n \otimes (J_q^-)^n)$$

$$= \sum_{n=0}^{\infty} A_n e^{2h(J_q^3 \otimes J_q^3)} (J_q^+ \otimes e^{-2hJ_q^3}) (J_q^+)^n \otimes (J_q^-)^n$$

$$+ \sum_{n=0}^{\infty} A_n e^{2h(J_q^3 \otimes J_q^3)} (J_q^+)^n \otimes J_q^+ (J_q^-)^n .$$

The lemma above and Lemma 3 in the appendix have been used. If we substitute the commutator from the lemma, this becomes

$$\sum_{n=0}^{\infty} A_n e^{2h(J_q^3 \otimes J_q^3)} (J_q^+ \otimes e^{-2hJ_q^3}) (J_q^+)^n \otimes (J_q^-)^n$$

$$+ \sum_{n=1}^{\infty} A_n e^{2h(J_q^3 \otimes J_q^3)} [n]_q [2J_q^3 + n - 1]_q (J_q^+)^n \otimes (J_q^-)^{n-1}$$

$$+ \sum_{n=0}^{\infty} A_n e^{2h(J_q^3 \otimes J_q^3)} (J_q^+)^n \otimes (J_q^-)^n J_q^+ .$$

On the other hand, the right-hand side of the equation is given by

$$\sum_{n=0}^{\infty} A_n e^{2h(J_q^3 \otimes J_q^3)} (J_q^+)^n \otimes (J_q^-)^n (J_q^+ \otimes e^{2hJ_q^3} + 1 \otimes J_q^+)$$

$$= \sum_{n=0}^{\infty} A_n e^{2h(J_q^3 \otimes J_q^3)} (J_q^+)^{n+1} \otimes (J_q^-)^n e^{2hJ_q^3} + \sum_{n=0}^{\infty} A_n e^{2h(J_q^3 \otimes J_q^3)} (J_q^+)^n \otimes (J_q^-)^n J_q^+$$

$$= \sum_{n=0}^{\infty} A_n e^{2h(J_q^3 \otimes J_q^3)} (J_q^+)^{n+1} \otimes q^{2n} e^{2hJ_q^3} (J_q^-)^n + \sum_{n=0}^{\infty} A_n e^{2h(J_q^3 \otimes J_q^3)} (J_q^+)^n \otimes (J_q^-)^n J_q^+ .$$

If we equate both sides, a very simple expression results, namely

$$\sum_{n=0}^{\infty} A_n e^{2h(J_q^3\otimes J_q^3)}(J_q^+ \otimes e^{-2hJ_q^3})(J_q^+)^n \otimes (J_q^-)^n$$
$$+\sum_{n=1}^{\infty} A_n e^{2h(J_q^3\otimes J_q^3)}[n]_q[2J_q^3 + n - 1]_q (J_q^+)^n \otimes (J_q^-)^n$$
$$= \sum_{n=0}^{\infty} A_n e^{2h(J_q^3\otimes J_q^3)}(1 \otimes q^{2n} e^{2hJ_q^3})(J_q^+)^{n+1} \otimes (J_q^-)^n .$$

Changing the limit on the second sum, and putting all terms onto the left-hand side, the following equation results:

$$\sum_{n=0}^{\infty}(A_n e^{2h(J_q^3\otimes J_q^3)}(1 \otimes e^{-2hJ_q^3} - 1 \otimes e^{2hJ_q^3} q^{2n})$$
$$+ A_{n+1} e^{2h(J_q^3\otimes J_q^3)}[n+1]_q[2J_q^3 + n]_q)(J_q^+)^{n+1} \otimes (J_q^-)^n = 0 .$$

In a slightly different form, this is

$$\sum_{n=0}^{\infty} e^{2h(J_q^3\otimes J_q^3)} \Big(A_n q^n (1 \otimes (q^{-n} e^{-2hJ_q^3} - q^n e^{2hJ_q^3})) + A_{n+1} \cdot [n+1]_q$$
$$\cdot \left(1 \otimes \frac{q^n q^{2hJ_q^3} - q^{-n} q^{-2hJ_q^3}}{q - q^{-1}}\right)\Big) \cdot (J_q^+)^{n+1} \otimes (J_q^-)^n .$$

This equation holds precisely when the following equation is satisfied:

$$A_n q^n = A_{n+1}[n+1]_q (q - q^{-1})^{-1} .$$

This is proved in Lemma 4.

To prove (5.2) and (5.3), the following theorem is required.

THEOREM

$$\Delta(J_q^+)^n = \sum_{r=0}^{n} e^{-r(n-r)h} \begin{bmatrix} n \\ r \end{bmatrix}_q (J_q^+)^r \otimes e^{2rhJ_q^3}(J_q^+)^{n-r} ,$$

$$\Delta(J_q^-)^n = \sum_{r=0}^{n} e^{-r(n-r)h} \begin{bmatrix} n \\ r \end{bmatrix}_q e^{-2(n-r)hJ_q^3}(J_q^-)^r \otimes (J_q^-)^{n-r} ,$$

$$\Delta(J_q^3)^n = \sum_{r=0}^{n} \binom{n}{r} (J_q^3)^r \otimes (J_q^3)^{n-r} .$$

These can be proved by induction on n. Only the second identity will be shown.

Using (2) and the homomorphism property of Δ, one has

$$\Delta(J_q^-)^k = \Delta(J_q^-)\Delta(J_q^-)^{k-1}$$
$$= (J_q^- \otimes 1 + e^{-2hJ_q^3} \otimes J_q^-) \sum_{r=0}^{k-1} e^{-r(k-1-r)h} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q$$
$$\cdot\, e^{-2(k-1-r)hJ_q^3} (J_q^-)^r \otimes (J_q^-)^{k-1-r}$$
$$= \sum_{r=0}^{k-1} e^{-r(k-1-r)h} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q J^- e^{-2(k-1-r)hJ_q^3} (J_q^-)^r \otimes (J_q^-)^{k-1-r}$$
$$+ \sum_{r=0}^{k-1} e^{-r(k-1-r)h} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q e^{-2(k-r)hJ_q^3} (J_q^-)^r \otimes (J_q^-)^{k-r}.$$

Using the identity

$$e^{2(k-1-r)hJ_q^3} J_q^- e^{-2(k-1-r)hJ_q^3} = J_q^- e^{-2(k-1-r)h},$$

the equation above becomes

$$\sum_{r=0}^{k-1} e^{-(r+2)(k-1-r)h} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q e^{-2(k-1-r)hJ_q^3} (J_q^-)^{r+1} \otimes (J_q^-)^{k-1-r}$$
$$+ \sum_{r=0}^{k-1} e^{-r(k-1-r)h} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q e^{-2(k-r)hJ_q^3} (J_q^-)^r \otimes (J_q^-)^{k-r}$$
$$= \sum_{r=1}^{k} e^{-(r+1)(k-r)h} \begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_q e^{-2(k-r)hJ_q^3} (J_q^-)^r \otimes (J_q^-)^{k-r}$$
$$+ e^{-2khJ_q^3} \otimes (J_q^-)^k + \sum_{r=1}^{k-1} e^{-r(k-1-r)h} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q e^{-2(k-r)hJ_q^3} (J_q^-)^r \otimes (J_q^-)^{k-r}$$
$$= \sum_{r=1}^{k-1} e^{-r(k-r)h-(k-r)h+(k-r)h} e^{-(k-r)h} \begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_q e^{-2(k-r)hJ_q^3} (J_q^-)^r \otimes (J_q^-)^{k-r} + (J_q^-)^k \otimes 1$$
$$+ e^{-2khJ_q^3} \otimes (J_q^-)^k + \sum_{r=1}^{k-1} e^{-r(k-1-r)h-rh} e^{rh} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q e^{-2(k-r)hJ_q^3} (J_q^-)^r \otimes (J_q^-)^{k-r}$$
$$= (J_q^-)^k \otimes 1 + \sum_{r=1}^{k-1} e^{-r(k-r)h} (e^{-(k-r)h} \begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_q$$
$$+ e^{rh} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q) e^{-2(k-r)hJ_q^3} (J_q^-)^r \otimes (J_q^-)^{k-r} + e^{-2khJ_q^3} \otimes (J_q^-)^k.$$

Using the identity

$$\begin{bmatrix} r \\ k \end{bmatrix}_q = q^{-k} \begin{bmatrix} r-1 \\ k \end{bmatrix}_q + q^{r-k} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q ,$$

which follows from the usual q-binomial coefficient identity, Lemma 5, by replacing q with q^{-1} and using $[n]_q = [n]_{q^{-1}}$, the equation above takes the form

$$(J_q^-)^k \otimes 1 + \sum_{r=1}^{k-1} e^{-r(k-r)h} \begin{bmatrix} k \\ r \end{bmatrix}_q e^{-2(k-r)hJ_q^3} (J_q^-)^r \otimes (J_q^-)^{k-r} + e^{-2khJ_q^3} \otimes (J_q^-)^k$$
$$= \sum_{r=0}^{k} e^{-r(k-r)h} \begin{bmatrix} k \\ r \end{bmatrix}_q e^{-2(k-r)hJ_q^3} (J_q^-)^r \otimes (J_q^-)^{k-r} .$$

It is required to prove that

$$(\Delta \otimes id)R = R_{13} R_{23} .$$

Here, R is the R-matrix given in (8). To evaluate the left-hand side, we use the identity for $\Delta(J_q^+)^k$ given above, then with $n = m + r$,

$$(\Delta \otimes id)R = \sum_n A_n (\Delta \otimes id\, e^{2h(J_q^3 \otimes J_q^3)}) \sum_{r=0}^{n} e^{-r(n-r)h} \begin{bmatrix} n \\ r \end{bmatrix}_q (J_q^+)^r \otimes e^{2rhJ_r^3} (J_q^+)^{n-r} \otimes (J_q^-)^n$$
$$= e^{2h(J_q^3 \otimes 1 \otimes J_q^3 + 1 \otimes J_q^3 \otimes J_q^3)} \sum_{m,r} A_{m+r} q^{-rm} \begin{bmatrix} m+r \\ m \end{bmatrix}_q (1 \otimes q^{2rJ_q^3} \otimes 1)((J_q^+)^r$$
$$\otimes (J_q^+)^m \otimes (J_q^-)^{m+r}) .$$

Using the definitions

$$R_{13} = e^{2h(J_q^3 \otimes 1 \otimes J_q^3)} \sum_{n=0}^{\infty} A_n (J_q^+)^n \otimes 1 \otimes (J_q^-)^n ,$$

$$R_{23} = e^{2h(1 \otimes J_q^3 \otimes J_q^3)} \sum_{n=0}^{\infty} A_n (1 \otimes (J_q^+)^n \otimes (J_q^-)^n) ,$$

the product $R_{13} \cdot R_{23}$ is given by

$$e^{2h(J_q^3 \otimes 1 \otimes J_q^3)} \sum_{m,n} A_n (J_q^+)^n \otimes 1 \otimes (J_q^-)^n e^{2h(1 \otimes J_q^3 \otimes J_q^3)} A_m (1 \otimes (J_q^+)^m \otimes (J_q^-)^m)$$
$$= e^{2h(J_q^3 \otimes 1 \otimes J_q^3 + 1 \otimes J_q^3 \otimes J_q^3)} \sum_{n,m} A_n A_m (1 \otimes q^{2nJ_q^3} \otimes 1)((J_q^+)^n \otimes (J_q^+)^m \otimes (J_q^-)^{n+m}) .$$

To show that both sides of (5.2) are equal, it then suffices to show that

$$q^{-mn} \begin{bmatrix} n+m \\ m \end{bmatrix} A_{m+n} = A_m A_n .$$

This is just Lemma 4.2.

Finally to verify the identity

$$(id \otimes \Delta)R = R_{13}R_{12},$$

one requires the identity for $\Delta(J_q^-)^n$, then with $n = m + r$,

$$\begin{aligned}(id \otimes \Delta)R &= \sum_n A_n (id \otimes \Delta) e^{2h(J_q^3 \otimes J_q^3)} \sum_r q^{-rm} \begin{bmatrix} m+r \\ r \end{bmatrix}_q (J_q^+)^n \otimes q^{-2mJ_q^3} (J_q^-)^r \otimes (J_q^-)^m \\ &= \sum_{m,r} A_{m+r} q^{-rm} (id \otimes \Delta) e^{2h(J_q^3 \otimes J_q^3)} \begin{bmatrix} m+r \\ r \end{bmatrix}_q (J_q^+)^{m+r} \otimes q^{-2mJ_q^3} (J_q^-)^r \otimes (J_q^-)^m \\ &= e^{2h(J_q^3 \otimes J_q^3 \otimes 1 + J_q^3 \otimes 1 \otimes J_q^3)} \sum_{m,r} A_{m+r} q^{-rm} \begin{bmatrix} m+r \\ r \end{bmatrix}_q (1 \otimes q^{-2mJ_q^3} \otimes 1)((J_q^+)^{m+r} \\ &\quad \otimes (J_q^-)^r \otimes (J_q^-)^m).\end{aligned}$$

Recall that

$$R_{13} = e^{2h(J_q^3 \otimes 1 \otimes J_q^3)} \sum A_n (J_q^+)^n \otimes 1 \otimes (J_q^-)^n$$

and

$$R_{12} = e^{2h(J_q^3 \otimes J_q^3 \otimes 1)} \sum A_m (J_q^+)^m \otimes (J_q^-)^m \otimes 1.$$

Then, the right-hand side is given by the expression

$$\begin{aligned}& e^{2h(J_q^3 \otimes 1 \otimes J_q^3)} \sum A_n (J_q^+)^n \otimes 1 \otimes (J_q^-)^n e^{2h(J_q^3 \otimes J_q^3 \otimes 1)} \sum A_m (J_q^+)^m \otimes (J_q^-)^m \otimes 1 \\ &= e^{2h(J_q^3 \otimes 1 \otimes J_q^3 + J_q^3 \otimes J_q^3 \otimes 1)} \sum_{n,m} A_n A_m (1 \otimes q^{-2nJ_q^3} \otimes 1) \cdot (J_q^+)^{n+m} \otimes (J_q^-)^m \otimes (J_q^-)^n.\end{aligned}$$

Both sides are equal again by Lemma 4.

4. Physical application-q-rotator model

In addition to the deep algebraic structure presented here, it is worthwhile to show that the quantum group $SU_q(2)$ has a useful physical application. Consider the q-rotator model, which is an exactly solvable system. This is the q deformation of the usual rigid rotator model with the quantum group symmetry $SU_q(2)$. To treat this system, it is necessary to write down the Hamiltonian for the system [7]. This Hamiltonian is given as follows:

$$H_{q,rot} = \hbar^2 \frac{C_{I,q}}{2I}, \tag{13}$$

where $C_{I,q}$ is the Casimir operator for the quantum group $SU_q(2)$:

$$C_{I,q} = J_q^- J_q^+ + [J_q^3]_q + [J_q^3 + 1]_q. \tag{14}$$

Also, the $J_q^{\pm}$, J_q^3 are generators of $SU_q(2)$ and can be realized in terms of generators of the SU(2) in the following way:

$$J_q^+ = \left(\frac{[J^3 + j]_q[J^3 - 1 - j]_q}{(J^3 + j)(J^3 - 1 - j)}\right)^{1/2} J^+ ,$$

$$J_q^- = J^- \left(\frac{[J^3 + j]_q[J^3 - 1 - j]_q}{(J^3 + j)(J^3 - 1 - j)}\right)^{1/2}$$

$$J_q^3 = J^3 ,$$

where j is an operator formally expressed as follows:

$$j = -\frac{1}{2} + \sinh^{-1}\left(\frac{\sinh\gamma}{\gamma}(C_{I,q} + \left[\frac{1}{2}\right]_q^2)^{1/2}\right).$$

When q is not a root of unity, it is easy to see that the representation of $SU_q(2)$ may be those of SU(2) up to a phase. If q is a root of unity, the approach must be different. Consequently, the representations, in coordinates, of $SU_q(2)$ can be chosen as spherical harmonics,

$$\tilde{\psi}_{JM}(\overrightarrow{x}) = Y_{JM}(\theta, \rho) .$$

This means the representations of the quantum group $SU_q(2)$ are completely reducible, while the action of the Casimir operator $C_{I,q}$ yields

$$C_{I,q}\tilde{\psi}_{JM} = [J + 1]_q[J]_q\tilde{\psi}_{JM} .$$

The q-rotator model is typically applied to diatomic molecules, and the results here will be especially relevant to molecules consisting of unlike atoms such that there is a significant internal dipole moment. By considering the dipole transition matrix elements [10], it is found from the orthogonality of the spherical harmonics that the matrix elements vanish unless $\Delta J = \pm 1$. The selection rule of the emission (absorption) of the q-rotator model is then $\Delta J = \pm 1$. The emission absorption spectrum is given by

$$\Delta\nu = \frac{E_{q,rot}(J + 1) - E_{q,rot}(J)}{hc} = B([J + 1]_q[J + 2]_q - [J + 1]_q[J]_q) ,$$

where $B = h/8\pi^2 Ic$. Using the definition of $[x]_q$, this can be written in a way which is more suitable for calculations:

$$[J + 1]_q([J + 2]_q - [J]_q) = \frac{q^{2(J+1)} - q^{-2(J+1)}}{q - q^{-1}} = \frac{\sinh 2\gamma(J + 1)}{\sinh\gamma} .$$

Substituting, one has

Table 1
A fit for the Lyman 0–2 R band in the H_2 molecule using equation $\nu_0 = 81153.35$, $H = -2259.29$ and $q = 2.91187$. The starred numbers have been interpolated.

J	ν	ν_{exp}	$\mid \nu - \nu_{exp} \mid$
0	82155.24	82155.24	0
1	82949.33	82125.75	823.6
2	83370.93	82026.75*	1344.2
3	83332.60	81861.90*	1470.7
4	82842.29	81631.51*	1210.8
5	82001.69	81337.61*	664.1
6	80985.14	80985.14	0
7	80003.48	80572.64	569.2
8	79260.30	80102.64*	842.3
9	78909.75	79588.80	679.1
10	79024.53	79024.53	0

$$\Delta\nu = B\frac{\sinh(2\gamma(J+1))}{\sinh\gamma} = H\sin h(2\gamma(J+1)) = A\sin(2g(J+1))\,, \tag{15}$$

where $A = iH$. By fitting the parameters g and A accordingly, values for the emission-absorption spectrum can be calculated and compared to the corresponding experimental values.

Two fits have been performed, first on the molecule H_2, and then on the molecule HCl, (Tables 1 and 2). The molecule H_2 is symmetric, and distorted largely by rotation and the dipole moment should play less of a role than in the molecule HCl, which has a rather strong moment. Consequently, we would expect eq. (15) to work most consistently on HCl. In spite of this, the numbers for H_2 in Table 1 provide a surprisingly good fit to the R-branch of the 0–2 band. The numbers obtained from

Table 2
A fit for the emission (absorption) spectrum of HCl using $\nu = H\sin(2g(J+1))$ with the values $H = 967.49$ (cm^{-1}) and $g = 0.010744$. The experimental numbers are from Herzberg [10].

J	ν	ν_{exp}	$\mid \nu - \nu_{exp} \mid$
0	20.787		
1	41.566		
2	62.325		
3	88.055	83.03	0.03
4	103.747	104.1	0.35
5	124.391	124.3	0.91
6	144.977	145.03	0.53
7	165.497	165.51	0.01
8	185.940	185.86	0.08
9	206.298	206.06	0.08
10	226.560	226.50	0.06

(15) for the molecule HCl provide a much more consistent fit, and are relatively accurate. The experimental numbers for H_2 are from the paper by Dabrowski [11] and the experimental numbers for HCl are found in Herzberg [12].

Appendix

LEMMA 1

$$[J_q^3, (J_q^{\pm})^n] = \pm n (J_q^{\pm})^n .$$

The proof is by induction on n. Taking the equation above as the induction hypothesis, one has

$$\begin{aligned}[J_q^3, (J_q^+)^n] &= J_q^3 (J_q^+)^n - (J_q^+)^n J_q^3 \\ &= J_q^3 (J_q^+)^{n-1} J_q^+ - (J_q^+)^{n-1} J_q^3 (J_q^+) + (J_q^+)^{n-1} J_q^3 J_q^+ - (J_q^+)^n J_q^3 \\ &= (n-1)(J_q^+)^n + (J_q^+)^{n-1} J_q^+ = n (J_q^+)^n .\end{aligned}$$

LEMMA 2

$$[J_q^+, (J_q^-)^n] = [n]_q [2J_q^3 + n - 1]_q (J_q^-)^{n-1} .$$

The proof is by induction on n. Suppose this equation holds up to $n-1$. Consider

$$\begin{aligned}[J_q^+, (J_q^-)^n] &= [J_q^+, (J_q^-)^{n-1}] J_q^- + (J_q^-)^{n-1} [J_q^+, J_q^-] \\ &= [n-1]_q [2J_q^3 + n - 2]_q (J_q^-)^{n-1} + (J_q^-)^{n-1} [2J_q^3]_q .\end{aligned}$$

Since

$$[2J_q^3]_q = \frac{q^{2J_q^3} - q^{-2J_q^3}}{q - q^{-1}}$$

and using

$$(J_q^-)^{n-1} q^{2J_q^3} = q^{2(n-1)} q^{2J_q^3} (J_q^-)^{n-1} , \; (J_q^-)^{n-1} q^{-2J_q^3} = q^{-2(n-1)} q^{2J_q^3} (J_q^-)^{n-1} ,$$

and simplifying, one obtains

$$\begin{aligned}[J_q^+, (J_q^-)^n] &= \frac{q^{2J_q^3 + 2n - 1} - q^{-2J_q^3 - 1} - q^{-2J_q^3 + 1} + q^{-2J_q^3 - 2n + 1}}{(q - q^{-1})(q - q^{-1})} (J_q^-)^{n-1} \\ &= [n]_q [2J_q^3 + n - 1]_q (J_q^-)^{n-1} .\end{aligned}$$

LEMMA 3

The following commutators are required to use in the first lemma, so that the exponentials can be permuted. First,

$$[J_q^3 \otimes J_q^3, J_q^+ \otimes 1] = [J_q^3, J_q^+] \otimes J_q^3 = J_q^+ \otimes J_q^3 = (J_q^+ \otimes 1)(1 \otimes J_q^3) .$$

This implies that

$$e^{-2h(J_q^3 \otimes J_q^3)}(J_q^+ \otimes 1)e^{2h(J_q^3 \otimes J_q^3)} = (J_q^+ \otimes 1)e^{-2h(1 \otimes J_q^3)} = (J_q^+ \otimes 1) \cdot (1 \otimes e^{-2hJ_q^3}) .$$

Similarly,

$$[J_q^3 \otimes J_q^3, e^{2hJ_q^3} \otimes J_q^+] = J_q^3 e^{2hJ_q^3}[J_q^3, J_q^+] = (e^{2hJ_q^3} \otimes J_q^+) \cdot (J_q^3 \otimes 1) .$$

This implies that

$$e^{-2h(J_q^3 \otimes J_q^3)}(e^{2hJ_q^3} \otimes J_q^+)e^{2h(J_q^3 \otimes J_q^3)} = 1 \otimes J_q^+ .$$

LEMMA 4.1

$$A_n q^n = A_{n+1}[n+1]_q (q - q^{-1})^{-1} .$$

Expanding, one obtains

$$A_{n+1}[n+1]_q (q-q^{-1})^{-1} = \frac{q^{(n+1)(n+2)/2}(1-q^{-2})^{n+1}}{[n+1]_q!}[n+1]_q(q-q^{-1})^{-1} = A_n q^n .$$

LEMMA 4.2

$$q^{-mn}\begin{bmatrix} n+m \\ m \end{bmatrix} A_{m+n} = A_m A_n .$$

Substituting for A_{m+n} on the left-hand side,

$$q^{-mn}\frac{[n+m]_q!}{[n]_q![m]_q!}\frac{q^{(m+n)(m+n+1)/2}(1-q^{-2})^{m+n}}{[m+n]_q!} = q^{(m^2+n^2+m+n)/2}\frac{(1-q^{-2})^{m+n}}{[n]_q![m]_q!}$$

$$= A_m A_n .$$

LEMMA 5

$$\begin{bmatrix} r \\ k \end{bmatrix}_q = q^{-k}\begin{bmatrix} r-1 \\ k \end{bmatrix}_q + q^{r-k}\begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q , \quad r \geqslant k \geqslant 0 .$$

By a direct calculation

$$\begin{aligned}\begin{bmatrix} r \\ k \end{bmatrix}_q - q^{-k}\begin{bmatrix} r-1 \\ k \end{bmatrix}_q &= \frac{[r-1]_q!}{[k]_q![r-k]_q!}\left(\frac{(q^r - q^{-r}) - q^{-k}(q^{r-k} - q^{-r+k})}{q - q^{-1}}\right) \\ &= q^{r-k}\frac{[r-1]_q!}{[k]_q![r-k]_q!}\frac{q^k - q^{-k}}{q - q^{-1}} \\ &= q^{r-k}\begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q .\end{aligned}$$

Notice that if one replaces q by q^{-1} and then uses $[k]_{q^{-1}} = [k]_q$ one obtains the result

$$\begin{bmatrix} r \\ k \end{bmatrix}_q = q^k\begin{bmatrix} r-1 \\ k \end{bmatrix}_q + q^{-(r-k)}\begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q .$$

References

[1] R. Barbier, J. Meyer and M. Kibler, J. Phys. G20 (1994) L13.
[2] R. Barbier, J. Meyer and M. Kibler, Int. J. Mod. Phys. 4 (1995) 385;
R. Barbier and M. Kibler, in: *Modern Group Theoretical Methods in Physics*, eds. J. Bertrand et al. (Kluwer Academic Publ., 1995) pp. 27–36.
[3] V.G. Drinfeld, *Proc. Int. Cong. Math. Amer. Math. Soc.*, Providence, RI (1987) p. 798.
[4] M. Kibler and T. Negadi, J. Math. Chem. 11 (1992) 13.
[5] R.B. Zhang, M.D. Gould and A.J. Bracken, Comm. Math. Phys. 137 (1991) 13.
[6] M. Jimbo, Int. J. Mod. Phys. A4 (1989) 3759.
[7] Z. Chang and H. Yen, Phys. Rev. A43 (1991) 6043.
[8] M. Sweedler, *Hopf Algebra* (W.A. Benjamin, New York, 1969).
[9] A.J. Bracken, D.S. McAnally, R.B. Zhang and M.D. Gould, J. Phys. A: Math. Gen. 24 (1991) 1379.
[10] L.D. Landau and E.M Lifshitz, *Quantum Mechanics*, 3rd ed. (Pergamon Press, 1977).
[11] I. Dabrowski, Can. J. Phys. 62 (1984) 1639.
[12] G. Herzberg, *Molecular Spectra and Molecular Structure, I: Spectra of Diatomic Molecules*, 2nd ed. (Van Nostrand, Princeton, 1955).